Exam 4

Q:1 ⓐ $\int (4x + 6\sqrt{x})\, dx$

$= 4\int x\, dx + 6\int x^{1/2}\, dx$

$= (4)\left(\frac{x^2}{2}\right) + 6\left(\frac{x^{3/2}}{3/2}\right) + C$

$= \boxed{2x^2 + 4x^{3/2} + C}$

ⓑ $\int \left(3\sec^2 x - \frac{2}{x}\right) dx$

$= 3\int \sec^2 x\, dx - 2\int \frac{dx}{x}$

$= \boxed{3\tan(x) - 2\ln|x| + C}$

Q:2

Given that

$$f'(x) = 8x^3 - 2x \quad , \quad f(1) = 7$$

Now

$$\int f'(x)\,dx = \int (8x^3 - 2x)\,dx$$

$$\hookrightarrow \quad f(x) = 8\int x^3\,dx - 2\int x\,dx$$

$$\Rightarrow \quad f(x) = (8)\left(\frac{x^4}{4}\right) - 2\left(\frac{x^2}{2}\right) + C$$

$$\Rightarrow \quad f(x) = 2x^4 - x^2 + C$$

Using $f(1) = 7 \Rightarrow f(1) = 2(1)^4 - (1)^2 + C$

$$\Rightarrow \quad 7 = 2 - 1 + C$$

$$\hookrightarrow \quad C = 6$$

So

$$\boxed{f(x) = 2x^4 - x^2 + 6}$$

Q:3 $a(t) = v'(t) = g = -9.8 \ m/s^2$

(a) $v(t) = \int v'(t)\,dt$

$\Rightarrow v(t) = \int (-9.8)\,dt$

$\Rightarrow v(t) = -9.8t + C_1$

Initial velocity is $v(0) = 30 \ m/s$

$\Rightarrow v(0) = -9.8(0) + C_1 \Rightarrow C = 30$

So $\boxed{v(t) = -9.8t + 30}$

(b) Position is $s(t) = \int v(t)\,dt$

$\hookrightarrow s = \int (-9.8t + 30)\,dt$

$= \frac{-9.8t^2}{2} + 30t + C \Rightarrow s = -4.9t^2 + 30t + C$

As $s(0) = 100 \Rightarrow s(0) = -4.9(0)^2 + 30(0) + C \Rightarrow C = 100$

So $\boxed{s(t) = -4.9t^2 + 30t + 100}$

(c) Maximum height occurs where velocity $= 0$

$\Rightarrow -9.8t + 30 = 0 \Rightarrow t = \frac{30}{9.8} \Rightarrow t \approx 3.1 \ sec$

So Maximum height $= s(3.1) = -4.9(3.1)^2 + 30(3.1) + 100$

$\hookrightarrow$ $\boxed{\text{Maximum Height} = 145.9 \text{ meters}}$

Q:4 $f(x) = x^2+3$. interval is $[0,6]$

$\Delta x = \frac{6-0}{3} \Rightarrow \Delta x = 2$

So $x_0 = 0,\ x_1 = 2,\ x_2 = 4,\ x_3 = 6$

(a) Left-endpoint approximation:

$$Area = (\Delta x)(f(x_0)+f(x_1)+f(x_2))$$

$$A = (2)(f(0)+f(2)+f(4))$$

$$A = 2((0)^2+3+(2)^2+3+(4)^2+3) = 2(3+4+3+16+3)$$

$$\Rightarrow Area = 2(29) \Rightarrow \boxed{Area = 58 \text{ unit}^2}$$

(b) Right endpoint approximation:

$$Area = (\Delta x)(f(x_1)+f(x_2)+f(x_3))$$

$$A = (2)(f(2)+f(4)+f(6))$$

$$A = (2)((2)^2+3+(4)^2+3+(6)^2+3) = 2(4+3+16+3+36+3)$$

$$A = 2(65) \Rightarrow \boxed{Area = 130 \text{ unit}^2}$$

BONUS Midpoint approximation

$$Area = \Delta x(f(1)+f(3)+f(5))$$

$$= 2((1)^2+3+(3)^2+3+(5)^2+3)$$

$$= 2(1+3+9+3+25+3)$$

$$= 2(44)$$

$$\boxed{Area = 88 \text{ unit}^2}$$

Q:5 Since $\int_2^{12} f(x)\,dx$ gives area under curve of $f(x)$ for $2 \le x \le 12$

From given diagram:

$$\int_2^{12} f(x) = \int_2^{6} |f(x)|dx + \int_6^{12} |f(x)|dx$$

Since For $2 \le x \le 6$, $f(x) < 0$ So

$$\Rightarrow \int_2^{12} f(x)dx = -\int_2^{6} f(x)dx + \int_6^{12} f(x)\,dx$$

Using Geometry

$$\int_2^{12} f(x)dx = -\left(\begin{array}{l}\text{Area of triangle} \\ \text{with base} = 4 \\ \text{and height} = 2\end{array}\right) + \begin{array}{l}\text{Area of} \\ \text{half circle} \\ \text{of radius} = 3\end{array}$$

$$= -\frac{1}{2}(4)(2) + \frac{1}{2}\pi(3)^2$$

$$\boxed{\int_2^{12} f(x)dx = -4 + \frac{9\pi}{2}}$$

Q:6

(a) $\int_0^4 (7f(x)+2g(x))\,dx$

$= 7\int_0^4 f(x)dx + 2\int_0^4 g(x)dx$

$= (7)(10)+(2)(-7) = 70-14$

$\boxed{\int_0^4 [7f(x)+2g(x)]\,dx = 56}$

(b) $\int_2^4 (f(x)+g(x))\,dx$

$= \int_2^0 f(x)dx + \int_0^4 f(x)dx$

$+ \int_2^0 g(x)dx + \int_0^4 f(x)dx$

$= -\int_0^2 f(x)dx + \int_0^4 f(x)dx - \int_0^2 g(x)dx + \int_0^4 f(x)dx$

$= -(-3) + 10 - (4) + (-7)$

$= 3+10-4-7$

$\boxed{\int_2^4 [f(x)+g(x)]\,dx = 2}$

Q:7

$$F(x) = \int_{x}^{x^2} \sec(t)\,dt$$

$$\Rightarrow \quad F(x) = \int_{x}^{0} \sec(t)\,dt + \int_{0}^{x^2} \sec(t)\,dt$$

$$\Rightarrow \quad F(x) = -\int_{0}^{x} \sec(t)\,dt + \int_{0}^{x^2} \sec(t)\,dt$$

Now

$$F'(x) = \frac{d}{dx}\left[-\int_{0}^{x} \sec(t)\,dt\right] + \frac{d}{dx}\left[\int_{0}^{x^2} \sec(t)\,dt\right]$$

Using Fundamental theorem of Calculus I

$$F'(x) = -(\sec(x))\frac{d}{dx}(x) + \sec(x^2)\cdot\frac{d}{dx}(x^2)$$

$$\boxed{F'(x) = -\sec(x) + 2x\sec(x^2)}$$

Q:8 (a)

$$\int_0^3 (3x^2-2)\,dx = 3\int_0^3 x^2\,dx - 2\int_0^3 dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^3 - 2\left[x\right]_0^3$$

$$= \left((3)^3-(0)^3\right) - 2(3-0)$$

$$= 27-0-6 \Rightarrow \boxed{\int_0^3 (3x^2-2)\,dx = 21}$$

(b)

$$\int_0^{\pi/3} \cos(x)\,dx = \left[\sin(x)\right]_0^{\pi/3}$$

$$= \sin\left(\frac{\pi}{3}\right) - \sin(0)$$

$$= \frac{\sqrt{3}}{2} - 0$$

$$\boxed{\int_0^{\frac{\pi}{3}} \cos(x)\,dx = \frac{\sqrt{3}}{2}}$$

BONUS

$$\text{Average Value} = \frac{1}{3-0}\int_0^3 (3x^2-2)\,dx$$

$$= \left(\frac{1}{3}\right)(21)$$

$$\boxed{f_{average} = 7}$$

Q:9 ⓐ $v(t) = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8)$

$\Rightarrow v(t) = 3(t-2)(t-4)$

Cyclist moves in positive direction when $v(t) > 0$

$\Rightarrow 3(t-2)(t-4) > 0$

$\Rightarrow t-2>0$ and $t-4>0$ | $t-2<0$ and $t-4<0$

$\Rightarrow t>2$ and $t>4$ | $\Rightarrow t<2$ and $t<4$

$\Rightarrow t>4$ | $\Rightarrow t<2$

So positive direction when $\boxed{t \in (0,2)}$

and So negative direction when

$v(t) < 0 \Rightarrow \boxed{t \in (2,3)}$

ⓑ Net distance $= \int_0^3 v(t)\,dt = \int_0^2 v(t)\,dt + \int_2^3 v(t)\,dt$

$\Rightarrow$ distance net $= \int_0^2 (3t^2 - 18t + 24)\,dt + \int_2^3 (3t^2 - 18t + 24)\,dt$

$= \left[t^3 - 9t^2 + 24t\right]_0^2 + \left[t^3 - 9t^2 + 24t\right]_2^3$

net distance $= \left((2)^3 - 9(2)^2 + 24(2)\right) - (0) + \left((3)^3 - 9(3)^2 + 24(3)\right) - \left((2)^3 - 9(2)^2 + 24(2)\right)$

$= 8 - 36 + 48 + 27 - 81 + 72 - 8 + 36 - 48$

$\boxed{\text{Net distance} = 18}$

ⓒ Total distance $= \int_0^3 |v(t)|\,dt$

Total distance $= + \int_0^2 (3t^2 - 18t + 24)\,dt - \int_2^3 (3t^2 - 18t + 24)\,dt$

$= \left[t^3 - 9t^2 + 24t\right]_0^2 - \left[t^3 - 9t^2 + 24t\right]_2^3$

$= (2)^3 - 9(2)^2 + 24(2) - 0 - \left((3)^3 - 9(3)^2 + 24(3) - (2)^3 + 9(2)^2 - 24(2)\right)$

$= 8 - 36 + 48 - 27 - 81 + 72 + 8 - 36 + 48$

$\boxed{\text{Total distance} = 4}$

Q:10

$$v(t) = \int a(t)\,dt$$

$$v(t) = \int (6t-8)\,dt$$

$$v(t) = 3t^2 - 8t + C_1$$

As $V(0) = 2$ $\Rightarrow V(0) = 3(0)^2 - 8(0) + C_1$

$\hookrightarrow 2 = C_1$

So $\boxed{v(t) = 3t^2 - 8t + 2 \quad m/s}$

Position function is:

$$S(t) = \int v(t)\,dt$$

$$\Rightarrow S(t) = \int (3t^2 - 8t + 2)\,dt$$

$$= 3\int t^2\,dt - 8\int t\,dt + 2\int dt$$

$$= t^3 - 4t^2 + 2t + C_2$$

Using $S(0) = 0$ $\Rightarrow S(0) = (0)^3 - 4(0)^2 + 2(0) + C_2$

$\Rightarrow 0 = C_2$

So $\boxed{S(t) = t^3 - 4t^2 + 2t}$

Q:11

(a) $\int \sqrt{3x+2}\, dx$

Let $u = 3x+2 \Rightarrow du = 3dx$

$\hookrightarrow \frac{1}{3} du = dx$

So $\int (3x+2)^{1/2} dx = \int u^{1/2} \cdot \frac{1}{3} du$

$= \left(\frac{1}{3}\right)\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{2}{9} u^{3/2} + C$

$\hookrightarrow \boxed{\int \sqrt{3x+2}\, dx = \frac{2}{9}(3x+2)^{3/2} + C}$

(b) $\int (1+x)(x-1)^{50} dx$

Let $u = x-1 \Rightarrow x = u+1$

$\hookrightarrow dx = du$

So $\int (1+x)(x-1)^{50} dx = \int (1+u+1)(u)^{50} du$

$= \int (u+2)\, u^{50}\, du$

$= \int u^{51} du + 2\int u^{50} du$

$= \frac{u^{52}}{52} + \frac{2u^{51}}{51} + C$

$\hookrightarrow \boxed{\int (1+x)(x-1)^{50} dx = \frac{(x-1)^{52}}{52} + \frac{2(x-1)^{51}}{51} + C}$

Q:12 $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{\sin^2(x)}\,dx$

Let $u = \sin(x) \Rightarrow du = \cos(x)\,dx$

when $x = \frac{\pi}{4} \Rightarrow u = \sin(\frac{\pi}{4}) \Rightarrow u = \frac{\sqrt{2}}{2}$

when $x = \frac{\pi}{2} \Rightarrow u = \sin(\frac{\pi}{2}) \Rightarrow u = 1$

So

$$\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{\sin^2(x)}\,dx = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{du}{u^2}$$

$$= \left[-\frac{1}{u}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= -\left[\frac{1}{1} - \frac{1}{\sqrt{2}/2}\right]$$

$$= -1 + \frac{2}{\sqrt{2}}$$

$$= -1 + \frac{(\sqrt{2})(\sqrt{2})}{\sqrt{2}}$$

$$\boxed{\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{\sin^2(x)}\,dx = -1 + \sqrt{2}}$$

Q:13 (a) $\int \frac{x+2}{x^2+4x-3}\,dx$

Let $u = x^2+4x-3 \Rightarrow \frac{du}{dx} = 2x+4$

$\hookrightarrow du = 2(x+2)\,dx \Rightarrow \frac{1}{2}du = (x+2)\,dx$

So $\int \frac{(x+2)}{x^2+4x-3}\,dx = \int \frac{(1/2)\,du}{u} = \frac{1}{2}\ln|u| + C$

$$\hookrightarrow \boxed{\int \frac{x+2}{x^2+4x-3}\,dx = \frac{1}{2}\ln|x^2+4x-3| + C}$$

(b) $\int_0^1 xe^{x^2}\,dx$

Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$x = 0 \Rightarrow u = (0)^2 \Rightarrow u = 0$

$x = 1 \Rightarrow u = (1)^2 \Rightarrow u = 1$

So

$$\int_0^1 xe^{x^2}\,dx = \int_0^1 e^u \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}\left[e^u\right]_0^1$$

$$= \frac{1}{2}\left(e^1 - e^0\right)$$

$$\hookrightarrow \boxed{\int_0^1 xe^{x^2}\,dx = \frac{e-1}{2}}$$

Q:14 (a) $\int \frac{1}{64+49x^2}\,dx = \int \frac{1}{64+(7x)^2}\,dx$

Let $u = 7x \Rightarrow du = 7dx \Rightarrow \frac{1}{7}du = dx$

So $\int \frac{1}{64+(7x)^2}\,dx = \int \frac{1}{64+u^2}\cdot\frac{1}{7}du = \left(\frac{1}{7}\right)\int \frac{du}{(8)^2+u^2}$

$= \left(\frac{1}{7}\right)\left(\frac{1}{8}\tan^{-1}\left(\frac{u}{8}\right)\right)+C$

$\hookrightarrow \boxed{\int \frac{1}{64+49x^2}\,dx = \frac{1}{56}\tan^{-1}\left(\frac{7x}{8}\right)+C}$

(b) $\int \frac{e^x}{\sqrt{e^{2x}-9}}\,dx = \int \frac{e^x}{\sqrt{(e^x)^2-9}}\,dx$

Let $u = e^x \Rightarrow du = e^x\,dx$

So $\int \frac{e^x}{\sqrt{(e^x)^2-9}}\,dx = \int \frac{du}{\sqrt{u^2-(3)^2}}$

$= \cosh^{-1}\left(\frac{u}{3}\right)+C$

$\hookrightarrow \boxed{\int \frac{e^x}{\sqrt{e^{2x}-9}}\,dx = \cosh^{-1}\left(\frac{e^x}{3}\right)+C}$